Market’s gamma hedging absorption capability for barrier options

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I - Vanilla Options - Definitions and examples

Options are contracts where the holder has a right and the seller a liability. A Call option gives its holder the right to buy an asset by a certain date at a certain price. A Put option gives its holder the right to sell an asset by a certain date at a certain price. The price at which the holder of the option can buy (sell) the asset and at which the seller has to sell (buy) is called the exercise price or strike. The date at which the option rights expire is called the expiration or maturity date. If the holder can only exercise his right at the maturity of the option contract then the option is of the European type. If, on the contrary, the holder can exercise his right at any time before or at maturity, the option is of the American type.

The profit of the buyer of a Call option will increase as the underlying price is higher than the strike price.

Mathematically the Call option payoff is defined as:

\[
\text{Payoff} = \max(S_T - K, 0)
\]

Example: Payoff for a Call option with a strike at 110% of the underlying

where Spot is the asset price at expiration and K is the exercise price.
Similarly, the payoff of the Put option is defined as:

\[ \text{Payoff} = \max(K - S_T, 0) \]

Example: Payoff for a Put option with a strike at 110% of the underlying

Above, we didn't use the option's premium for the Payoff calculation.

**A - Calculation of the premium**

**1- Process followed by the underlying**

The underlying follows the following geometric Brownian Motion model:

\[ dS = \mu S dt + \sigma S dW_t \]

with \( W_t \) denoting a standard Brownian motion under \( \sigma \) and \( \mu \) fixed
**Ito’s lemma**: In its simplest form, Ito’s lemma states the following:

\[ dx = a(x, t) \, dt + b(x, t) \, dW_t \]

And any twice differentiable scalar function \( f(t, x) \) of two real variables \( t \) and \( x \), one has

\[ dG = \left[ \frac{\partial G}{\partial x} \cdot a + \frac{\partial G}{\partial t} + \frac{1}{2} \cdot \frac{\partial^2 G}{\partial x^2} \cdot b^2 \right] \, dt + \frac{\partial G}{\partial x} \cdot b \, dW_t \]

Because the process satisfies the stochastic differential equation, we can apply Ito’s Lemma:

\[ dG = \left[ \frac{\partial G}{\partial x} \cdot \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \cdot \frac{\partial^2 G}{\partial x^2} \cdot \sigma^2 S^2 \right] \, dt + \frac{\partial G}{\partial x} \cdot \sigma S \, dW_t \]

With the following variable change: \( G = \ln(S) \)

\[ \frac{\partial G}{\partial S} = \frac{1}{S} ; \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} ; \quad \frac{\partial G}{\partial t} = 0 \]

\[ d\ln(S) = \sigma dW + \left( \mu - \frac{\sigma^2}{2} \right) \, dt \]

We can conclude that the underlying price follows the process under:

\[ S_T = S_0 \cdot \exp\left[ \left( \mu - \frac{\sigma^2}{2} \right) \cdot T + \sigma W_t \right] \]

**2 - Call Premium**

At maturity, the price of a Call option can be defined as:

\[ \text{Call}_T = \text{Max} (S_T - K, 0) \]

Extension of the formula by adding the probabilities:

\[ \text{Call}_0 = E_Q [S_T - K] \cdot \exp(-r \cdot T) \]

\[ = (S_T - K) \cdot P_Q[S_T > K] \cdot \exp(-r \cdot T) + 0 \cdot P_Q[S_T \leq K] \cdot \exp(-rT) \]

\[ = S_T \cdot P_Q[S_T > K] \cdot \exp(-r \cdot T) - K \cdot P_Q[S_T > K] \cdot \exp(-r \cdot T) \]
How to calculate the probability that the underlying is over the Strike at maturity?

\[ S_T > K \iff S_0 \ast \exp \left[ \left( r - \frac{\sigma^2}{2} \right) \ast T + \sigma W_T \right] \geq K \]

\[ \iff \left[ \left( r - \frac{\sigma^2}{2} \right) \ast T + \sigma W_T \right] \geq \ln \left( \frac{K}{S} \right) \]

\[ \iff \left[ \left( r - \frac{\sigma^2}{2} \right) \ast T + \sigma \sqrt{T} \right] \geq \ln \left( \frac{K}{S} \right) \]

\[ \iff -u = \frac{\ln \left( \frac{S}{K} \right) - \left( r - \frac{\sigma^2}{2} \right) \ast T}{\sigma \sqrt{T}} = d_2 \]

\[ \text{Call}_0 = S_T \ast N(d_2) \ast \exp(-r \ast T) - K \ast \exp(-r \ast T) \ast N(d_2) \]

To make the formula easier to understand we can simplify the following expression:

\[ S_T \ast N(d_2) \ast \exp(-r \ast T) \]

\[ S_T \ast N(d_2) \ast \exp(-r \ast T) = S_0 \ast \exp \left[ \left( r - \frac{\sigma^2}{2} \right) \ast T + \sigma W_T \right] \ast N(d_2) \ast \exp(-r \ast T) \]

\[ = S_0 \ast \exp \left[ - \left( \frac{\sigma^2}{2} \right) \ast T \right] \ast \exp[\sigma \sqrt{T}] \ast \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} \ast \exp \left( - \frac{u^2}{2} \right) du \]

With the variable change \( v = u - \sigma \sqrt{T} \) we have the following relation:

\[ S_T \ast N(d_2) \ast \exp(-r \ast T) = S_0 \ast \int_{-\infty}^{d_2 + \sigma \sqrt{T}} \frac{1}{\sqrt{2\pi}} \ast \exp \left( - \frac{1}{2} \ast v^2 \right) dv \]

Noting \( d_1 = d_2 + \sigma \sqrt{T} \)

\[ S_T \ast N(d_2) \ast \exp(-r \ast T) = S_0 \ast N(d_1) \]

The premium of a Call is

\[ \text{Call}_0 = S_0 \ast N(d_1) - K \ast \exp(-r \ast T) \ast N(d_2) \]

By the same demonstration we have the premium of a Put:

\[ \text{Put}_0 = -S_0 \ast N(-d_1) + K \ast \exp(-r \ast T) \ast N(-d_2) \]
3 - Put-Call Parity

In financial mathematics, Put–Call parity defines a relationship between the price of a European Call option and European Put option, both with the identical strike price and expiry, namely that a portfolio of long a Call option and short a Put option is equivalent to (and hence has the same value as) a single forward contract at this strike price and expiry.

Mathematically, we can assume that the relation between a Call and a Put option is:

\[
Call_0 - Put_0 = S_0 * N(d_1) - K * \exp(-r * T) * N(d_2) - S_0 * N(-d_1) + K * \exp(-r * T) * N(-d_2)
\]

\[
Call_0 - Put_0 = S_0 * N(d_1) - K * \exp(-r * T) * N(d_2) - K * \exp(-r * T) + K * \exp(-r * T) * N(d_2)
+ S_0 - S_0 * N(d_1)
\]

\[
Call_0 - Put_0 = S_0 - K * \exp(-r * T)
\]

B - Greeks

In mathematics, the Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The name is used because the most common of these sensitivities are often denoted by Greeks letters.

1 - Delta:

Delta (\(\Delta\)) measures the rate of change of option value with respect to changes in the underlying asset’s price. Delta is the first derivatives of the value \(v\) of the option with respect to the underlying instrument’s price \(S\).

Delta-Call:

\[
\Delta = \frac{\partial C}{\partial S} = N(d_1) + S \cdot \frac{\partial N(d_1)}{\partial S} - K \cdot \exp(-r \cdot T) \cdot \frac{\partial N(d_2)}{\partial S}
\]

\[
\frac{\partial N(d_1)}{\partial S} = d_1^* \cdot \varphi(d_1) \quad \text{with} \quad \varphi(d_1) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{d_1^2}{2}\right)
\]

\[
\frac{\partial C}{\partial S} = N(d_1) + S \cdot d_1^* \cdot \varphi(d_1) - K \cdot \exp(-r \cdot T) \cdot d_2^* \cdot \varphi(d_2)
\]
We have \( d_1 = d_2 + \sigma \sqrt{T} \iff d_1' = d_2' \)

\[
\varphi(d_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2}{2}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_1 - \sigma \sqrt{T})^2}{2}\right) = \varphi(d_1) \exp(r \cdot T) \cdot \frac{S}{K}
\]

\[
\frac{\partial C}{\partial S} = N(d_1) + S \cdot d_1' \cdot \varphi(d_1) - K \cdot \exp(-r \cdot T) \cdot d_1' \cdot \varphi(d_1) \cdot \exp(r \cdot T) \cdot \frac{S}{K}
\]

\[
\frac{\partial C}{\partial S} = N(d_1) ; \quad \frac{\partial P}{\partial S} = -N(-d_1)
\]

2 - Gamma:

Gamma (\( \gamma \)) measures the rate of change in the delta with respect to changes in the underlying price. Gamma is the second derivative of the value function with respect to the underlying price. All long options have positive gamma and all short options have negative gamma. Gamma is the greatest approximately at-the-money (ATM) and diminishes the further out you go either in-the-money (ITM) or out-of-the-money (OTM). Gamma is important because it corrects for the convexity of value.

When a trader seeks to establish an effective delta-hedge for a portfolio, the trader may also seek to neutralize the portfolio’s gamma, as this will ensure that the hedge will be effective over a wider range of underlying price movements. However, in neutralizing the gamma of a portfolio, alpha (the return in excess of the risk-free-rate) is reduced.

Gamma-Call:

\[
\gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial N(d_1)}{\partial S} = d_1' \cdot \varphi(d_1) \quad \text{with} \quad d_1' = \frac{1}{\sigma \sqrt{T}} \cdot \frac{1}{S/K} \cdot \frac{1}{K} = \frac{1}{S \cdot \sigma \sqrt{T}}
\]

\[
\frac{\partial^2 C}{\partial S^2} = \frac{\varphi(d_1)}{S \cdot \sigma \sqrt{T}} = \frac{\partial^2 P}{\partial S^2}
\]
3 - Vega

Vega measures sensitivity to volatility. Vega is the derivative of the option value with respect to the volatility of the underlying asset.

Vega is not the name of any Greek letter. However, the glyph used is the Greek letter ν.

Vega can be an important Greek to monitor for an option trader, especially in volatile markets, since the value of some option strategies can be particularly sensitive to changes in volatility. The value of an option straddle, for example, is extremely dependent on changes to volatility.

\[
\frac{\partial C}{\partial \sigma} = S \cdot d_1 \cdot \varphi(d_1) - K \cdot \exp(-r \cdot T) \cdot (d_1' - \sqrt{T}) \cdot \varphi(d_1') \cdot \exp(r \cdot T) \cdot \frac{S}{K}
\]

Simplifying the expression hereabove, we can conclude that

\[
\frac{\partial C}{\partial \sigma} = 0S \cdot \sqrt{T} \cdot \varphi(d_1) = \frac{\partial P}{\partial \sigma}
\]

4 - Theta

Theta (θ) measures the sensitivity of the value of the derivative to the passage of time.

Theta is almost always negative for long Calls and Puts and positive for short (or written) Calls and Puts.

The value of an option can be analysed into two parts: the intrinsic value and the time value. The intrinsic value is the amount of money you would gain if you exercised the option immediately, so a Call with strike $50 on a stock with price $60 would have intrinsic value of $10, whereas the corresponding Put would have zero intrinsic value. The time value is the value of having the option of waiting longer before deciding to exercise. Even a deeply out of the money Put will be worth something, as there is some chance the stock price will fall below the strike before the expiry date. However, as time approaches maturity, there is less chance of this happening, so the time value of an option is decreasing with time. Thus if you
are long an option you are short theta: your portfolio will lose value with the passage of time (all other factors held constant).

\[
\frac{\partial C}{\partial T} = S \cdot d_1' \cdot \varphi(d_1) - K \cdot \exp(-r \cdot T) \cdot d_2' \cdot \varphi(d_2) + r \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2)
\]

\[
\theta_c = \frac{\partial C}{\partial T} = -r \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2) - S \cdot \frac{\sigma}{\sqrt{T}} \cdot \varphi(d_1)
\]

\[
\theta_p = \frac{\partial P}{\partial T} = r \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2) - S \cdot \frac{\sigma}{\sqrt{T}} \cdot \varphi(d_1)
\]

5 - Rho

Rho ($\rho$) measures sensitivity to the interest rate: it is the derivatives of the option value with respect to the risk free interest rate (for the relevant outstanding term). Except under extreme circumstances, the value of an option is less sensitive to changes in the risk free interest rate than to changes in other parameters. For this reason, rho is the least used of the first-order Greeks.

\[
\frac{\partial C}{\partial r} = S \cdot d_1' \cdot \varphi(d_1) - K \cdot \exp(-r \cdot T) \cdot d_2' \cdot \varphi(d_2) + T \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2)
\]

\[
= S \cdot d_1' \cdot \varphi(d_1) + T \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2) - K \cdot \exp(-r \cdot T) \cdot d_1' \cdot \varphi(d_1) \cdot \exp(r \cdot T) \cdot \left(\frac{S}{K}\right)
\]

\[
\rho_c = \frac{\partial C}{\partial r} = T \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2)
\]

\[
\rho_p = \frac{\partial P}{\partial r} = T \cdot K \cdot \exp(-r \cdot T) \cdot N(d_2)
\]

6 - Examples

Here are some examples of Payoffs and Greeks for Call and Put options. Calculations are made with our own pricer coded in Visual Basic.
The data used for those examples are:

- **Underlying price** = 100
- **Strike price** = 110
- **Risk-free rate** = 2.5%
- **Time to maturity** = 365 days
- **Volatility** = 25%

All the graphics show the quantity of a variable according to the underlying price.
C - Dynamic Delta Hedging

Hedging is the practice of making a portfolio of investments less sensitive to changes in market variables.

Delta neutral describes a portfolio of related financial securities, in which the portfolio value remains unchanged due to small changes in the value of the underlying security. Such a portfolio typically contains options and their corresponding underlying securities such that positive and negative delta components offset, resulting in the portfolio’s value being relatively insensitive to changes in the value of the underlying security.

A related term, delta hedging is the process of setting or keeping the delta of a portfolio as close to zero as possible. In practice, maintaining a zero delta is very complex because there are risks associated with re-hedging on large movements in the underlying stock’s price, and research indicates portfolios tend to have lower cash flows if re-hedged too frequently.

1 - Delta Neutral – How to make profit?

Delta Neutral Trading is capable of making a profit without taking any directional risk. This means that a delta neutral trading position can profit when the underlying stock stays stagnant or when the underlying stock rallies or ditches strongly. This is fulfilled in 4 ways:

1. By the bid ask spread of the option. This is a technique only option trading market makers can execute, which is simply buying at the bid price and simultaneously selling at the ask price, creating a net delta zero transaction and profiting from the bid/ask spread with no directional risk at all. This is also known as “Scalping”.

2. By time decay. When a position is delta neutral, having 0 delta value, it is not affected by small movements made by the underlying stock, but it is still affected by time decay as the premium value of the options involved continue to decay.

3. By volatility. By executing a delta neutral position, one can profit from a change in volatility with no directional risk when the underlying stock moves insignificantly. This option trading strategy is extremely useful when implied volatility is expected to change drastically soon.

4. By creating volatile option trading strategies. Even though delta neutral positions are not affected by small changes in the underlying stock, it can still profit from large, significant moves.
2 - Dynamic Hedging

Delta value in option trading changes all the time due to gamma value, moving a delta neutral trading position slowly out of its delta neutral state and into a directional biased state. Even though this behaviour allows delta neutral trading positions to profit in all directions, in a delta neutral position that is created in order to take advantage of volatility or time decay without any directional risk, the delta neutral state needs to be continuously maintained and “resetted”. This continuous resetting of an option trading position’s delta value to zero is Dynamic Delta Hedging or simply, Dynamic Hedging.

Mathematically a Delta Neutral portfolio should equals to:

\[ P_h = n_1 \times D_1 + n_2 \times D_2 = 0 \]

Where \( D_1 \) = Delta value of the original options.
\( D_2 \) = Delta value of the hedging options.
\( n_1 \) = Amount of original options.
\( n_2 \) = Amount of hedging options.

Below, we have simulated the underlying price via Monte Carlo simulations and we calculated the PnL of the position (long Call option).

The data used for the simulations are:

Spot price = 100
Strike price = 90
Volatility = 0.1
Risk-free rate = 0.05
Time to maturity = 1
Number of simulations = 100
Number of steps = 365
Monte Carlo methods are a broad class of computational algorithms that reply on repeated random sampling to obtain numerical results i.e. by running simulations many times (code in annex 1).

Above, there are 100 simulations of the underlying price in 365 periods. Below, the is the average price of the 100 simulations.
For the Dynamic Delta Hedging, we need the delta level at each step. We calculated the delta at each period with the Black-Scholes model.

As said before, the delta level rises up to 1 because the option is more and more in the money. Indeed, in our case, the underlying is globally growing.

The PnL of the hedging position is the following:

The method used for the calculation of the hedging portfolio PnL is on annex 2.
II - Relations between Digital options and Call Spread

A - Digital options

In finance, a binary option is a type of option where the payoff is either a fixed amount of asset or nothing at all. The two main kind of binary options are the cash-or-nothing binary option and the asset-or-nothing binary option. The cash-or-nothing binary option pays a fixed amount of cash if the option expires in-the-money while the asset-or-nothing pays the value of the underlying security.

If the underlying price is over the strike price, the options pays a certain amount of money to the holder of the option. If the underlying price is under the strike price, there is no gain.

Here is the payoff for a 10% cash-or-nothing binary option. When the spot price is over the strike price (Here 105), the holder earn 10% of the underlying price. There is also the payoff of an asset-or-nothing Call option:

B - How to approach a cash or northing option payoff?

We can approach a cash-or-nothing Call option by using a Call-Spread strategy.

The Call spread consist in buying a Call option and selling another Call option with an higher strike. Losses and gains in this strategies are limited. Indeed, if the underlying is under the lowest strike \((K_1)\), losses are limited to \((K_1 - K_2)\). if the underlying is over the highest strike \((K_2)\), gains are limited to \((K_2 - K_1)\). We realise in those two cases that the payoff of the Call
spread is similar to the digital option. However, the payoff is different when the spot price is between $K_1$ and $K_2$.

From this, we can approach a cash-or-nothing Call option by buying a Call spread with two close strikes. But as the strikes are closer, the payoff of the Call-spread Strategy is lower.

In the graph under, coded in Matlab, we can see four Call spread strategies with different strike prices. We notice that the Call spread 80-120 deliver 40 if the underlying price is over 120 while the Call spread 110-120 deliver 10 if the underlying is over 120.

We can conclude that we have to buy four Call-spread 110-120 to be delivered the same amount as a Call spread 80-120 in our example. The slope between $K_1$ and $K_2$ will be closer from the slope of the cash-or-nothing option.
The more strikes are close, the more we have to buy Call spreads to have the same amount of cash than a cash-or-nothing option.

With a Matlab code, we priced the payoff of 10 Call spread 100-101, and the payoff of one binary 10%. We can see that the two payoffs are quasi similar.

But in the market conditions, it is quasi impossible to find two Call options with strikes separated by only 1€ except on the OTC market.

To put ourselves in market conditions, we priced some payoffs of Call spread strategies and binary option with Bloomberg. The underlying is the Eurostoxx 50 (SX5E Index).

The first option we priced is a cash-or-nothing option. The strike price is 2775€. In order to make the results more significant, we bought 100,000 contracts.
We tried to approach the payoff of this cash-or-nothing option by buying Call spreads with different strike prices. As we said above, we realized that we had to buy more Call spreads to approach this strategy when the strikes were closer.

The first strategy was to buy Call spread with strikes separated by 75€. To approach the same payoff from the digital option we had to buy 152 Call spreads:

The second strategy was to buy Call spread with strikes separated by 50€. To approach the same payoff from the digital option we had to buy 212 Call spreads:
The last strategy was to buy Call spread with strikes separated by 25€. To approach the same payoff from the digital option we had to buy 452 Call spreads:

C - Delta and Gamma Call-Spread/Digital

We learnt just earlier that we could approach a binary Call Option via a Call-Spread strategy.

Now, we will compare the Delta and the Gamma of those two products.

1 - Delta

In those graphs, we can see that the delta of those two products have the same form.

The delta is the highest when the spot price is really close from the strike price. And more the spot price is distant from the strike, more the delta approach zero. Moreover, the delta is higher when the maturity approach.
2 - Gamma

As the delta, the gamma of those two products are quasi similar, they are switching in the same times. The more the maturity approach, the more the gamma is sensitive.

We can also see that the gamma is higher when we buy more Call spread options. The Indeed, we can see it in the next graph:

Annexes 3 and 4: Matlab codes used for those charts.
III - Barrier options

A - Definition

In finance, a barrier option is an exotic derivative option on the underlying asset whose price breaching the pre-set barrier level either springs the option into existence or extinguishes an already existing option.

Will the payoff of standard Call and Put options only depends on the price of the underlying at maturity, barrier options are path-dependent exotic derivatives whose value depends on the underlying having breached a given level, the barrier during a certain period of time. The market for barrier options has grown strongly because they are less expensive than standard options and provide a tool for risk managers to better express their markets views without paying for outcomes that they may find unlikely.

We can divide barrier options into knock-in and knock-out options. An European knock-in option is an option whose holder is entitled to receive a standard European option if a given level is breached before expiration date or a rebate otherwise. An European knock-out option us a standard European option that ceases to exist if the barrier is touched, giving its holder the right to receive a rebate. In both cases, the rebate can be zero.

The way in which the barrier is breached is important in the pricing of barrier options and, therefore, we can define down-and-in, up-and-in, down-and-out, up-and-out options for both calls and puts, giving us a total of eight different barrier options. There are more complex types of barrier options like double barrier options.

We can appreciate a vanilla Call option using barrier options. Indeed, suppose that we have a portfolio composed of a down-and-in Call and a down-and-out Call with identical characteristic and no rebate. If the barrier is never hit, the down-and-out Call provides us a standard Call. If the barrier is hit then the down-and-out Call expires worthless but the down-and-in Call emerges as a standard Call. Either way we end up with a vanilla Call so the following relationship between barrier options and vanilla options must hold when the rebate is zero.

With barrier options, investors can express more complex views than the simple bullish or bearish scenario that a vanilla Call will permit. Barrier option are generally cheaper for a
reason: hedge may vanish when most needed. Often, the sensitivity of barrier options to market will be very sharp, bigger than that of vanilla options.

In the graph under, we can see the payoff of an up-and-in Barrier Option with Matlab. The Barrier is fixed at a price of 120. We notice that if the Barrier is hit, the payoff become similar as vanilla Call option.

![Payoff - Barrier option - Up-and-in](image)

**B - Greeks**

The risk management of barrier options is very difficult, especially when the underlying is near the barrier and even more when there is very little time left.

Traders can use dynamic hedge (delta hedging) or static hedges (use other options).

**Examples:**

An up-and-out Call is cheaper than the standard Call. As the underlying moves up the vanilla Call will always gain in value but the barrier option will initially gain but then start to decline as the risk of getting knocked out will increase.

There are two competing effects - increase in value of the payoff, and the decrease in probability of receiving it. This makes the delta fluctuate and it can switch from positive to negative.
1 - Down and In Call

When the price of the underlying stock is high, the down-and-in Call can be worth a lot less than the Vanilla Call if barrier is far, because it will get knocked in only when the stock has fallen a lot. Below the barrier, it is the same as vanilla Call.

The value is the greatest when the spot price is close to the barrier.

Just below barrier, delta is the same as that of vanilla Call. Just above barrier, delta is negative - as stock price increases, the knock-in probability decreases. Big change from positive to negative delta at barrier means delta hedging is difficult (long to short stock as barrier crossed). Gamma always positive and very large at barrier.
2 - Down and In Put

Since much of the value of a standard Put is due to downward moves, which would also cause the down-and-in Put to be knocked in, both options can have similar value. If strike is below the barrier, it is worth the same as Vanilla Put.

Just above barrier, delta is far less than that of vanilla Put. The chance of knock-in is reduced when underlying price increases, leading to decline in option value.

Gamma is large and positive near barrier and infinite at the barrier. The Vega is also positive.
3 - Down and Out Call

Worth nearly the same as European Call if barrier is far, because it is knocked out when Call has left value left.

Below the barrier delta is zero (Call is worth 0); above the barrier Call price goes up fast. Delta higher than that of vanilla Call just above the barrier. Gamma is extremely large at the barrier. Option decreases as volatility increases.

Delta decreases as stock price increases just slightly higher than barrier.

When the underlying is very far from barrier, the barrier has little effect and value, delta gamma all are like that of the vanilla Call.
4 - Down and Out Put

The down-and-out Put will get knocked out when the stock has moved down and the European Put is in the money. Therefore it is priced much cheaper than standard Put. If strike is below the barrier it is worthless.

Just above the barrier, delta is large and positive- if price moves slightly higher the probability of knock-out is reduced. At much higher price levels, the barrier’s impact is less and delta is negative like that of the vanilla Put.

Gamma is very large and negative at the barrier. Also short Vega at the barrier.
5 - Up and In Call

If strike is above the barrier, the up-and-in Call is worth the same as a Vanilla Call.

It is not much cheaper than Vanilla Call because it is knocked in for large moves that contribute to value of a vanilla Call.

As barrier is approached, it has large gain in value and delta is greater than of vanilla Call, in some region even greater than. Gamma is very large for the same reason.
6 - Up and In Put

Up-and-in Put is much cheaper than the standard Put because it becomes alive only when the underlying has moved up and the standard Put is worth very little.

At the barrier, the option is equal in value to vanilla Put, and after knock-in, it declines with rising stock price just as a vanilla Put. The maximum value of the option is at the barrier.

Gamma is positive and extremely large at the barrier. Vega is also very large at the barrier.
7 - Up and Out Call

The up-and-out Call is much cheaper than the Vanilla Call, because it is knocked out precisely when large up moves occur - and these are the moves which will contribute to the vanilla Call’s value. If strike is above the barrier, this option is worth zero.

Much below barrier, delta is positive just like vanilla, but as barrier is approached, delta becomes negative; after barrier is crossed delta is zero.

Gamma is infinite at barrier but we can only plot the gamma slightly to left and slightly to the right of barrier. If the price of the underlying is just below the barrier, delta hedging is a killer because for a small up move gamma goes to zero.
Up-and-out Put would be similar to Vanilla Put, because when it is knocked out owing to large up moves, the vanilla option might be worth little anyway.

Below the barrier, delta is similar to vanilla Put but above the barrier it is zero as option is knocked out.

Likewise, gamma is similar to that of vanilla Put below the barrier and zero above - and it is infinite at the barrier.
Impact of a strike shift on long Call knock-in position:

<table>
<thead>
<tr>
<th>Strike</th>
<th>Price</th>
<th>Delta</th>
<th>Gamma</th>
<th>Vega</th>
<th>Thêta</th>
<th>Rho</th>
</tr>
</thead>
<tbody>
<tr>
<td>2675</td>
<td>20.04</td>
<td>32.01</td>
<td>11.64</td>
<td>5.55</td>
<td>-0.98</td>
<td>0.01</td>
</tr>
<tr>
<td>2700</td>
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<td>29.89</td>
<td>10.86</td>
<td>5.18</td>
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<td>0.01</td>
</tr>
<tr>
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<td>10.09</td>
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<td>0.01</td>
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<td>2750</td>
<td>16.07</td>
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<td>4.45</td>
<td>-0.78</td>
<td>0.01</td>
</tr>
<tr>
<td>2775</td>
<td>14.75</td>
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<td>8.54</td>
<td>4.08</td>
<td>-0.72</td>
<td>0.01</td>
</tr>
<tr>
<td>2800</td>
<td>13.43</td>
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<td>7.77</td>
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<td>0.01</td>
</tr>
<tr>
<td>2825</td>
<td>12.11</td>
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<td>3.36</td>
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<td>0.01</td>
</tr>
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<td>2850</td>
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<td>6.25</td>
<td>3.00</td>
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<td>0.00</td>
</tr>
<tr>
<td>2875</td>
<td>9.51</td>
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<tr>
<td>2925</td>
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<td>4.09</td>
<td>1.98</td>
<td>-0.34</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Sources: Bloomberg

C - How to approach a barrier option payoff?

We can approach barrier options by different ways. In our examples, we will try to approach an up-and-in barrier option.

The first way is to buy an asset-or-nothing Call option with a strike at the level of our barrier option. Indeed, when the strike is hit (here the level of our barrier), the gain is a certain amount and as the spot price is high as the gain is high.
Another way to approach an up-and-in barrier option is to buy a cash-or-nothing digital option and another Call with the same strike.

At the end, we can buy a Call spread with two close strike (to approach a digital option) and another Call with the same strike as the Call spread. The difference between this strategy and a barrier option is the spread between the highest strike of the call spread and the second call.
D - Gamma hedging

Gamma neutral hedging is the construction of options trading positions that are hedged such that the total gamma value of the position is zero or near zero, resulting in the delta value of the positions remaining stagnant no matter how strongly the underlying stock moves.

The problem with the delta neutral hedging (defined before - part I-C-1) is that even though it prevents the position from reacting to small changes in the underlying stock, it is still prone to sudden big moves which can take option traders off guard with no time to dynamically rebalance the position at all. This is where gamma neutral hedging comes in. By hedging an options trading position to gamma neutral, the position’s delta value is completely frozen and when used in conjunction with delta neutral position, the position’s delta value stays at 0 no matter how widely the underlying stock moves, thereby keeping the value of the position completely stagnant. Such a position is known as a delta neutral gamma neutral position.

1 - Reduction of position volatility

Gamma is another way of representing the amount of volatility of an options trading position. Big gamma values lead to big changes in delta value, resulting in exponential gains or losses. By going gamma neutral, position delta is fixed no matter how much the underlying stock moves, producing a highly predictable and calculable income based on the delta value. This is known as a delta positive, gamma neutral position.

2 - Sealing in profits

If a period of high volatility is to be expected and the options trading position has made a good profit so far, instead of sealing in the profits by selling the position, we could actually perform a delta neutral gamma neutral hedge to completely seal in the profits. A delta neutral gamma neutral position is not moved either by moves in the underlying stock nor time decay (gamma neutral positions are also automatically theta neutral). So, when implied volatility rises as the period of high volatility approaches, the position stands to gain in value due to rising volatility.
3 - Gamma Hedging can 'pin' a stock approaching expiry

As an investor who is long gamma can delta hedge by sitting on the bid and offer, this trade can pin an underlying to the strike. This is a side effect of selling if the stock rises above the strike, and buying if the stock falls below the strike. The amount of buying and selling has to be significant compared with the traded volume of the underlying, which is why pinning normally occurs for relatively illiquid stocks or where the position is particularly sizeable. Given the high trading volume of indices, it is difficult to pin a major index. Pinning is more likely to occur in relatively calm markets, where there is no strong trend to drive the stock away from its pin.

Pin risk occurs when the market price of the underlier of an option contract at the time of the contract’s expiration is close to the option’s strike price. In this situation, the underlier is said to have pinned. The risk to the seller of the option is that they cannot predict with certainty whether the option will be exercised or not. So the seller cannot hedge his position precisely and may end up with a loss or gain. There is a chance that the price of the underlier may move adversely, resulting in an unanticipated loss to the writer. In other words an option position may result in a large, undesired risky position in the underlier immediately after expiration, regardless of the actions of the seller.

**Example of a large size of Swisscom convertible pinned underlying for many months**

One of the most visible examples of pinning occurred in late 2004/early 2005, due to a large Swiss government debt issue, (Swisscom 0% 2005) convertible into the relatively illiquid Swisscom shares. As the shares traded close to the strike approaching maturity, the upward trend of the stock was broken. Swisscom was pinned for two to three months until the exchangeable expired. After expiration, the stock snapped back to where it would have been if the upward trend had not been paused. A similar event occurred to AXA in the month preceding the Jun05 expiry, when it was pinned close to €20 despite the broader market rising (after expiry AXA rose 4% in four days to make up for its earlier underperformance).
**E - Hedging barrier options**

Barrier options are difficult to hedge because they combine features of plain vanilla options and a bet on whether the underlying asset price hits the barrier. Merely adding a barrier provision to a regular option significantly impacts the sensitivity of the option value to changes in the price or the volatility of the underlying assets: The Greeks of barrier options behave very differently from those of plain vanilla options. The discontinuity in the payoff of barrier options complicates hedging, especially those that are in the money when they come out of existence (such as up and out calls and down and out puts). Delta hedging these options is particularly unpractical. For example, when the price nears the barrier and the option is about to expire, the Delta (and the Gamma) of an up and out Call takes large negative values because the option payoff turns into a spike in this region. Vega also turns negative when the price of the underlying asset is close to the barrier because a volatility pickup near the barrier increases the likelihood of the price passing through it. In contrast, the Delta, Gamma, and Vega of regular options are always positive and ‘well behaved’ functions.

Despite the difficulties associated with hedging barrier options, banks write large amounts of those instruments to accommodate customer demand and typically prefer to hedge their positions. Moreover, when devising hedging strategies, banks have to face the limitations of real markets, like transaction costs, discrete trading, and lack of liquidity. To compensate the associated loss of flexibility one may prefer to hedge barrier options with regular options.
instead of with the underlying asset and a riskless bond because regular options are more closely related to barrier options.

**F - Barrier Shift**

Sometimes, because of the ‘Greeks’ and the time left, traders have to sell an excess of shares below the barrier level. The result incurs a loss on the sale of the excess shares. To avoid this loss, the trader should give himself a cushion to sell any excess shares over the barrier. In order to do that, the trader prices and risk manages a slightly different option. Namely, an option where the barrier is shifted downward in such a way that the trader has enough room to sell the excess shares without incurring a loss. This means that, if the trader believes that he needs a 3% cushion to sell the shares over the barrier, the trader prices and risk manages a 100/67% down-and-in put rather than a 100/70% one. The following sub-section discusses the factors that influence the magnitude of the barrier shift. (a long down-and-in put position goes from being long gamma to short gamma as the share price approaches the barrier).

There are 5 main influencing factors that impact the magnitude of the barrier shift. (factors are for a dow-and-in put position):

- The size of the dow-and-in put transaction. The larger the size the more shares have to be sold over the barrier and therefore the trader is more likely to move the share price against him, i.e. downward. In other words, the larger the size the larger the barrier shift.

- The difference between strike price and barrier level. If this difference is large, the dow-and-in put goes from not being a put to a put that is far in the money when the share price breaches the barrier. This means that the greater the difference between strike price and barrier level the greater the barrier shift that a trader would apply.

- The volatility of the underlying stock. The larger the volatility of a stock the larger is the risk to the trader of the stock price approaching the barrier level. If a stock is or can be highly volatile, the buyer of a dow-and-in put needs a larger barrier shift in order to be protected against a larger move. In other words, the larger the volatility the larger the barrier shift.

- The barrier level. Since lower stock prices tend to go hand in hand with higher volatilities and higher volatilities result in larger barrier shifts, traders need to apply larger barrier shifts to lower barrier levels.
- The time to left maturity. The closer one gets to maturity the larger absolute delta will be just before the barrier and therefore the larger the change in delta over the barrier. This obviously translates into larger barrier shifts for shorter maturities and typically traders use a scattered barrier table where the barrier shift increases once the down-and-in put gets closer to maturity. The reason that, if the stock price is just above the barrier, the absolute delta becomes larger when the time left to maturity gets shorter is because there is less time left for the down-and-in put to knock in.

Impact of a barrier shift on a long Call knock-in position:

<table>
<thead>
<tr>
<th>Barrier Shift</th>
<th>Price</th>
<th>Delta</th>
<th>Gamma</th>
<th>Vega</th>
<th>Thêta</th>
<th>Rho</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2%</td>
<td>15.07</td>
<td>23.94</td>
<td>8.633</td>
<td>4.13</td>
<td>-0.73</td>
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<tr>
<td>0%</td>
<td>14.75</td>
<td>23.53</td>
<td>8.542</td>
<td>4.08</td>
<td>-0.72</td>
<td>0.01</td>
</tr>
<tr>
<td>2%</td>
<td>14.43</td>
<td>23.13</td>
<td>8.449</td>
<td>4.04</td>
<td>-0.71</td>
<td>0.01</td>
</tr>
<tr>
<td>4%</td>
<td>14.11</td>
<td>22.72</td>
<td>8.355</td>
<td>4.00</td>
<td>-0.70</td>
<td>0.01</td>
</tr>
<tr>
<td>6%</td>
<td>13.80</td>
<td>22.33</td>
<td>8.2613</td>
<td>3.95</td>
<td>-0.70</td>
<td>0.01</td>
</tr>
<tr>
<td>8%</td>
<td>13.50</td>
<td>21.93</td>
<td>8.166</td>
<td>3.91</td>
<td>-0.69</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Sources: Bloomberg

The strategy used is to be long of a knock-in barrier with the following data:
Spot price = 2781€ Maturity = 06/21/2013
Strike price = 2775€ Volatility = 16.064%
Barrier = 3059.71€

We can realize in this chart that the delta is more affected by a barrier shift than the gamma. Indeed, with a 8% shift, the delta decrease of 6.8% while the gamma decrease of 2.2%.
Annexes

1 - Matlab code for Monte Carlo simulations

```matlab
function SPaths=MonteCarlo(S0,mu,sigma,T,NSteps,NRepl)
dt = T/NSteps;
nudt = (mu-0.5*sigma^2)*dt;
sidt = sigma*sqrt(dt);
Increments = nudt + sidt*randn(NRepl,NSteps);
LogPaths = cumsum([log(S0)*ones(NRepl,1),Increments],2);
SPaths = exp(LogPaths);
Spaths(:,1)=S0;
end
```

2 - Matlab code for calculation of a delta hedged option PnL

```matlab
function [ProfitLoss] = PnL(K,cours,NSteps,T,sigma,mu)
ProfitLoss = zeros(length(cours),1);
Time = zeros(length(cours),1);
Price = zeros(length(cours),1);
Ppos = zeros(length(cours),1);
Pneg = zeros(length(cours),1);
Hedgeerror = zeros(length(cours),1);
Cashflow = zeros(length(cours),1);
Balance = zeros(length(cours),1);
Netposition = zeros(length(cours),1);
dt = T / NSteps;
Time(1,1)=T;
for i=2:NSteps
    Time(i,1) = Time(i-1,1) - dt;
end
d1 = zeros(NSteps,1);```
d2 = zeros(NSteps,1);
for i=1:NSteps
    d1(i,1) = (log(cours(i,1)/K)+(mu+.5*sigma^2)*Time(i,1))/(sigma*sqrt(Time(i,1)));
    d2(i,1) = d1(i,1)-sigma*sqrt(Time(i,1));
end
delta = zeros(NSteps,1);
for i=1:NSteps
    delta(i,1) = normcdf(d1(i,1),0,1);
end
for i=1:NSteps
    Price(i,1) = cours(i,1)*normcdf(d1(i,1),0,1)-K*exp(-mu*Time(i,1))*normcdf(d2(i,1),0,1);
end
Pneg(1,1) = 0;
Ppos(1,1) = Price(1,1)-delta(1,1)*cours(1,1);
Hedgeerror(1,1) = 0;
Cashflow(1,1) = 0;
Balance (1,1) = -(delta(1,1)*cours(1,1))-Price(1,1);
Netposition(1,1) = 0;
for i=2 : NSteps;
    Pneg(i,1) = Price(i,1)-delta(i-1,1)*cours(i,1);
Ppos(i,1) = Price(i,1)-delta(i,1)*cours(i,1);
    Hedgeerror(i,1) = Ppos(i-1,1)*exp(mu*dt)-Pneg(i,1);
Cashflow(i,1) = (delta(i,1)-delta(i-1,1))*cours(i,1);
    Balance(i,1) = Balance(i-1,1)*exp(mu*dt)+Cashflow(i,1);
    Netposition(i,1) = Balance(i,1)+(Price(i,1)-delta(i,1)*cours(i,1));
end
ProfitLoss = Netposition;
OR

```matlab
function [PLBSCH, PLANA] = PLBS(cours, sigma, T, NSteps, mu, K)

dt = T/NSteps;
TimeLeft = zeros(NSteps,1);
TimeLeft(1,1) = T;
for i=2:NSteps
    TimeLeft(i,1) = TimeLeft(i-1,1)-dt;
end

PLBSCH = zeros(NSteps,1);
PLANA = zeros(NSteps,1);
calldelta = zeros(NSteps,1);
callprice = zeros(NSteps,1);
for i = 1:NSteps
    d1 = (log(cours(i,1)/K)+(mu+.5*sigma^2)*TimeLeft(i,1))/(sigma*sqrt(TimeLeft(i,1)));
    d2 = d1 - sigma*sqrt(TimeLeft(i,1));
    gamma = normpdf(d1,0,1)/(cours(i,1)*sigma*sqrt(TimeLeft(i,1)));
    Brownian = randn*sqrt(TimeLeft(i,1));
    process = mu*dt + sigma * Brownian;
    process2 = process^2;
    v = (1/dt) * (process2);
    PLBSCH(i,1) = -0.5*(cours(i,1)^2)*gamma*(v-sigma^2)*dt;
    callprice(i,1) = cours(i,1)*normcdf(d1,0,1)-K*exp(mu*TimeLeft(i,1))*normcdf(d2,0,1);
    calldelta(i,1) = normcdf(d1,0,1);
end

len = length(cours);
callpl=zeros(len,1);
callpl(1)=0;
for i=2:len
    callpl(i) = exp(-mu*dt*(i-1))*(callprice(i)-callprice(i-1)-calldelta(i-1)*(cours(i,1)-cours(i-1,1))-(exp(mu*dt)-1)*(callprice(i-1)-calldelta(i-1)*cours(i-1,1)));
end

PLANA = cumsum(callpl);
end
```
Cours = linspace(0.5*S,1.5*S,100)';
DigitalP = zeros(length(Cours),4);
DigitalD = zeros(length(Cours),4);
DigitalG = zeros(length(Cours),4);
Time = [0 0.25 0.50 0.99]';
for j=1:4
    for i=1:length(Cours)
        d1 = (log((Cours(i,1))/K)+(r+0.5*(vol*vol))*(T-Time(j,1)))/(vol*sqrt(T-Time(j,1)));
        d2 = d1 - vol*sqrt(T-Time(j,1));
        DigitalP(i,j) = exp(-r*(T-Time(j,1)))*normcdf(d1,0,1);
        DigitalD(i,j) = (exp(-r*(T-Time(j,1)))*normpdf(d2,0,1))/(vol*Cours(i,1)*sqrt(T-Time(j,1)));
        DigitalG(i,j) = -((exp(-r*(T-Time(j,1)))*d1*normpdf(d2,0,1))/(vol^2*Cours(i,1)^2*(T-Time(j,1))));
    end
end
SSJ = Cours;
end
4 - Matlab code for Greeks of a call spread strategy

```matlab
function [CPD CPG SSJ] = CPC(S,K,vol,r,T,Nb,width)
Cours = linspace(0.5*S,1.5*S,100)';
CPD = zeros(length(Cours),4);
CPG = zeros(length(Cours),4);
Time = [0 0.25 0.50 0.99]';
l = (width*S)/2;

for j=1:4
    for i=1:length(Cours)
        d11 = (log((Cours(i,1))/(K-l))+(r+0.5*(vol^2))*(T-Time(j,1)))/(vol*sqrt(T-Time(j,1)));
        d21 = d11 - vol*sqrt(T-Time(j,1));
        d12 = (log((Cours(i,1))/(K+l))+(r+0.5*(vol^2))*(T-Time(j,1)))/(vol*sqrt(T-Time(j,1)));
        d22 = d12 - vol*sqrt(T-Time(j,1));
        CPD(i,j) = Nb*(normcdf(d11))-Nb*((normcdf(d12)));
        CPG(i,j) = Nb*((normpdf(d11,0,1))/(vol*Cours(i,1)*sqrt(T-Time(j,1))))-Nb*((normpdf(d12,0,1))/(vol*Cours(i,1)*sqrt(T-Time(j,1))));
    end
end
SSJ = Cours;
end
```
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