Skyrmion configurations for Near-BPS Skyrme Models

N. Giasson, and L. Marleau

Département de physique, génie physique et d'optique
Université Laval, Québec, Canada

Introduction

The Skyrme Model, which was first proposed over half a century ago, is now understood as a low-energy effective theory of QCD where baryons (and nuclei) emerge as topological solitons. It provides a relatively good picture of the nucleons, but it overestimates their binding energies in nuclei by at least an order of magnitude. The present work is based on a recent extension of the original Skyrme model, the so-called near-BPS Skyrme Model. In the pure BPS limit of the model, there exists an infinite dimensional family of solutions with zero binding energy.

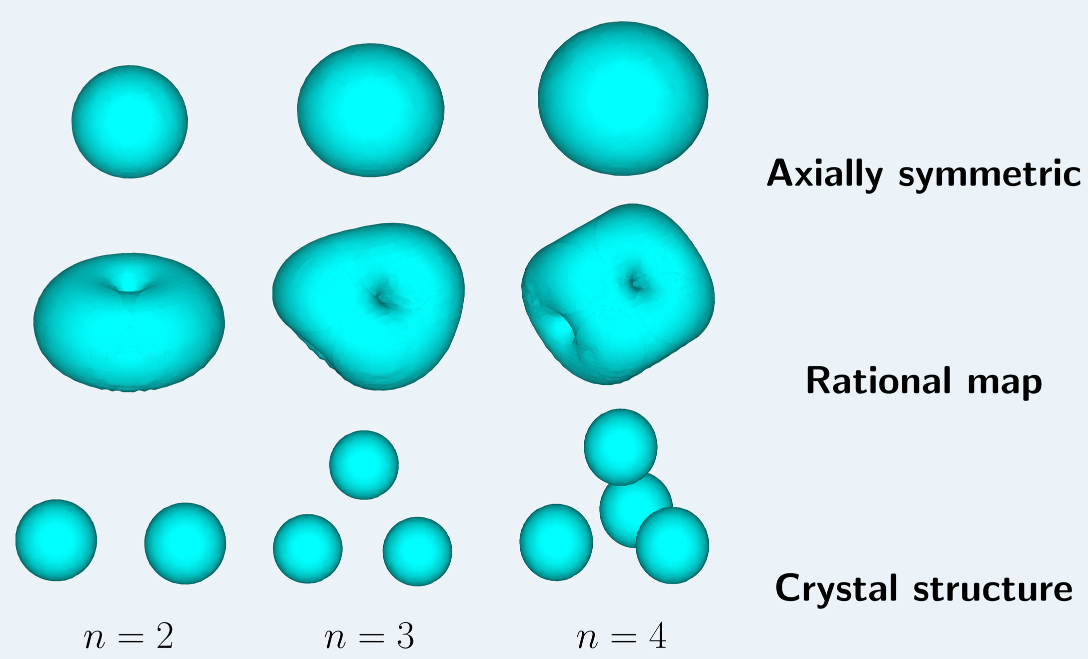
On the other hand, the solutions that arise from the near-BPS model nearly saturate the Bogomol'nyi bound which means that by construction there must have small binding energies. However, there remains an open question: what solution configuration minimizes the energy? Numerical calculations run into problems when one attempts to provide such an answer.

Here, instead, we compare the two most prominent configurations, the axial and rational map ansatz, for a class of hybrid model that goes from the original to the pure BPS Skyrme model. Our results suggest that the axial solution is the lowest energy configurations for the set of parameters for which the binding energy per nucleon B$/n$ agrees best with the experimental data thereby supporting even further the idea that nuclei could be near-BPS Skyrmions.

Motivation

One of the most attractive ideas to describe low-energy QCD physics is the Skyrme Model[1], an effective meson (pion) field theory motivated by arguments such as $1/N_c$ expansion or more recently holographic QCD[2]. Although the model is constructed out of meson fields alone, it can describe baryons and nuclei: they arise as topological solitons whose winding number corresponds to the baryon number n. Indeed, the Skyrme Model:

- Provides an effective meson field theory motivated by low-energy QCD.
- Describes pion and baryon physics within at least 30% accuracy.
- But, fails at multibaryon physics or nuclei.
- Binding energies too large especially for small nuclei.
- Finding lowest energy configurations is numerically challenging, but some clever parametrization of the fields equations can be used to obtain exact or approximated results without heavy numerical computation. The most commonly used are:



- Not all solutions are consistent with constant densities observed in nuclei.
- Further analyses of various potential (mass) terms, rotational deformations, higher order terms in derivatives or additional mesons (e.g. ω, ρ, ...) leads to similar configurations and binding energies.

Hint: The mass of a nucleus is almost proportional to that of the nucleon, $M_{\text{nuclei}} \approx n \cdot M_{\text{nucleon}}$, and since BPS-solitons follows this exact pattern, we propose a Skyrme-like model in a regime where the solutions are **Near-BPS solitons**.

Near-BPS Skyrme Models

We consider an extension of the original Skyrme Model with the Lagrangian density

$$\mathcal{L}_{\text{NBPS}} = \underbrace{\mathcal{L}_0 + \mathcal{L}_6}_{\text{BPS-solitons}} + \underbrace{\mathcal{L}_2 + \mathcal{L}_4}_{\text{Skyrme}} \tag{1}$$

such that solutions remain close to saturation of Bogomol'nyi bound without losing the link with the Skyrme Model. The pion fields ϕ_i are introduced through the $SU(2)$ matrix $U = \phi_0 + i\tau_i\phi_i$ (τ_i is the set of Pauli matrices) with the non-linear condition $\phi_0^2 + \phi_i^2 = 1$. Then using $L_\mu = U^\dagger \partial_\mu U$, we have

- $\mathcal{L}_0 = -\mu^2 V(U)$: the potential term
- $\mathcal{L}_2 = -\alpha \text{Tr}(L_\mu L^\mu)$: the NLσ term, quadratic in field derivatives.
- $\mathcal{L}_4 = +\beta \text{Tr}\left(\left[L_\mu, L_\nu\right]^2\right)$: the quartic Skyrme term (necessary to stabilize soliton in the Skyrme Model).
- $\mathcal{L}_6 = -\frac{3}{2}\frac{\lambda^2}{16^2}\text{Tr}\left(\left[L_\mu, L_\nu\right]\left[L^\nu, L^\rho\right]\left[L_\rho, L^\mu\right]\right)$: the sextic term which remains **quadratic** in time derivatives.

The Near-BPS approach consist in assuming that $\mathcal{L}_0 + \mathcal{L}_6$ dominate and treat $\mathcal{L}_2$ and $\mathcal{L}_4$ are small as perturbations. Finite energy solutions require a conserved topological charge which Skyrme identified as the baryon number or mass number n in the context of nuclei

$$n = \int d^3r \mathcal{B}^0 = -\frac{\epsilon^{ijk}}{24\pi^2}\int d^3r \text{Tr}\left(L_i L_j L_k\right). \tag{2}$$

When $\alpha, \beta = 0$, the solutions are BPS-solitons so their masses are exactly proportional to n. Here, two ansatzes will be considered for the parametrization of the pionic degrees of freedom (the U matrix).

First, we choose for simplicity an **axially symmetric** ansatz[3] (BPS solutions in the limit where $\alpha = \beta = 0$)

$$U = \cos(F(r))\mathbb{I}_2 + i\hat{\mathbf{n}} \cdot \tau \sin(F(r)), \tag{3}$$

Near-BPS Skyrme Models...

where $\mathbb{I}_2$ is the 2x2 identity matrix and $\hat{\mathbf{n}}$, a unit vector encoding the angular part of the solution, can be written in terms of the usual spherical coordinates θ and ϕ.

$$\hat{\mathbf{n}} = (\sin\theta\cos n\phi, \sin\theta\sin n\phi, \cos\theta). \tag{4}$$

Note that for $n > 1$, this is not the absolute static energy minimizer for $\alpha, \beta \neq 0$; they have been conjectured to emerge from "restricted harmonic map"[5].

On the other hand, the Skyrme Model ($\mathcal{L}_2 + \mathcal{L}_4$) leads to "rational map" solutions in which the profile F has no angular dependence. The ansatz (4) preserves this property and should anyhow provide a close upper bound of the static energy for small α, β. Considering these arguments, we then introduce the **rational map** ansatz in the context of the Near-BPS Skyrme Model. The general form reads

$$R(z) = \frac{p(z)}{q(z)}, \quad \bar{R}(\bar{z}) = \frac{\bar{p}(\bar{z})}{\bar{q}(\bar{z})}, \tag{5}$$

where $p(z)$ and $q(z)$ are complex polynomial of at most degree n. The first few polynomials that can be found in the literature for the original Skyrme model are summarize in the following table.

n	$R(z)$	$\mathcal{I}$	Sym.
1	z	1.00	$O(3)$
2	z^2	5.81	$D_{\infty h}$
3	$\frac{z^3-\sqrt{3}iz}{\sqrt{3}iz^2-1}$	13.58	T_d
4	$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$	20.65	O_h

This leads to a different expression for the unit vector $\hat{\mathbf{n}}$, that is

$$\hat{\mathbf{n}} = \frac{1}{1+R\bar{R}}\left(R+\bar{R},\ -i(R-\bar{R}),\ 1-R\bar{R}\right). \tag{6}$$

Substituting theses results into $\mathcal{L}_0$, $\mathcal{L}_2$, $\mathcal{L}_4$, $\mathcal{L}_6$, we can compute the static energy for these two distinct ansatzes. We are left with expressions of the form

$$E_0 = 4\pi\mu^2\int V(F)r^2dr \tag{7}$$

$$E_2 = 8\pi\alpha\int\left(F'^2r^2 + f_1\sin^2(F)\right)dr \tag{8}$$

$$E_4 = 64\pi\beta\int\left(f_1\sin^2(F)F'^2 + f_2\frac{\sin^4(F)}{r^2}\right)dr \tag{9}$$

$$E_6 = \frac{9\pi\lambda^2 f_2}{4}\int\frac{\sin^4(F)F'^2}{r^2}dr, \tag{10}$$

where

$$f_1 = \begin{cases} n^2+1, & \text{for Ax} \\ 2n, & \text{for RM} \end{cases}, \qquad f_2 = \begin{cases} n^2, & \text{for Ax} \\ \mathcal{I}, & \text{for RM} \end{cases}. \tag{11}$$

We can then write $E_{stat} = E_0 + E_2 + E_4 + E_6$, with Ax and RM the labels of the axially symmetric and rational map ansatz. Moreover, the angular integral $\mathcal{I}$ is traditionally define as

$$\mathcal{I} \equiv \frac{1}{4\pi}\int\left(\frac{(1+z\bar{z})^2}{(1+R\bar{R})^2}\frac{dR}{dz}\frac{d\bar{R}}{d\bar{z}}\right)^2\frac{2i}{(1+z\bar{z})^2}dzd\bar{z}. \tag{12}$$

Before going any further, it is important to point out that other energetic contributions, such as rotational energy (obtain from the quantization process), isospin energy and Coulomb energy can also be computed from these result.

However, since they do not account for a significant portion of the total mass of a nucleus, and that we wish to highlight the differences between the minimal energy configurations of the axially symmetric and rational map solutions for the **deuterium** ($n=2$) and **helium-4** ($n=4$) nuclei, these aforementioned energetic contributions will not be include here.

Energy minimization procedure

Considering E_{stat}, it turn out that one important ingredient is still missing in order to make an explicit computation of the static energy, the potential $V(F)$.
Moreover, we are interested in a model that can reproduce the **constant baryon density** in nuclei. Thus, we propose these few steps to obtain the minimal energy configurations**:**

Step 1: Introduce an appropriate $V(F)$ in order to generate constant baryon density. We found by inspection[4] that

$$V(F) = \frac{56\left(\sin^2\left(\frac{F}{2}\right)-1\right)^3\sin^2\left(\frac{F}{2}\right)\left(14\ln\left(\sin^2\left(\frac{F}{2}\right)\right)-5\right)}{45\left(\sqrt{25-70\ln\left(\sin^2\left(\frac{F}{2}\right)\right)}-5\right)} \tag{13}$$

obeys this criterion.

Step 2: We propose to write E_{stat} in terms of a new parameter ν (containing the μ,α,β,λ dependence) which introduces the relative weight of the BPS and Skyrme contributions to the Lagrangian:

$$E_{stat}(\nu) = (1-\nu)E_0 + \nu E_2 + \nu E_4 + (1-\nu)E_6, \tag{14}$$

where ν runs from 0 to 1 by steps of 0.05. Thus, we can adjust the relative strength of both sub-models when searching for the optimal configuration.

Step 3: We resort to a **simulated annealing** algorithm to minimize the static energy for the entire ν spectrum. That is, for each values of ν considered, the algorithm numerically compute the chiral angle F that minimizes E_{stat}, making uses of equations (7)-(10) with a given n value.

Step 4: The limit $\nu \to 0$ corresponds to the pure BPS model, which can be solved analytically for $F(r)$ in both cases. Therefore we have

$$F(r) = 2\arcsin\left(e^{-a^2r^2-\frac{7}{5}a^4r^4}\right), \quad a = \begin{cases} \left(\frac{\mu}{18n\lambda}\right)^{\frac{1}{3}}, & \text{for Ax} \\ \left(\frac{\mu}{18\sqrt{\mathcal{I}}\lambda}\right)^{\frac{1}{3}}, & \text{for RM} \end{cases}. \tag{15}$$

Results and analysis

In order to compare the results, we choose to compute energetic ratios for each values of ν so we define

$$\mathrm{R}_n^\nu = \frac{E_{stat}(\nu, n)}{E_{stat}(\nu, n=1)}. \tag{16}$$

For example, we can compute the ratios in the limit $\nu \to 0$, which gives

$$\mathrm{R}_n^0 = \begin{cases} n, & \text{for Ax} \\ \sqrt{\mathcal{I}}, & \text{for RM} \end{cases}. \tag{17}$$

The results for $n=2$ are shown in the following figure.

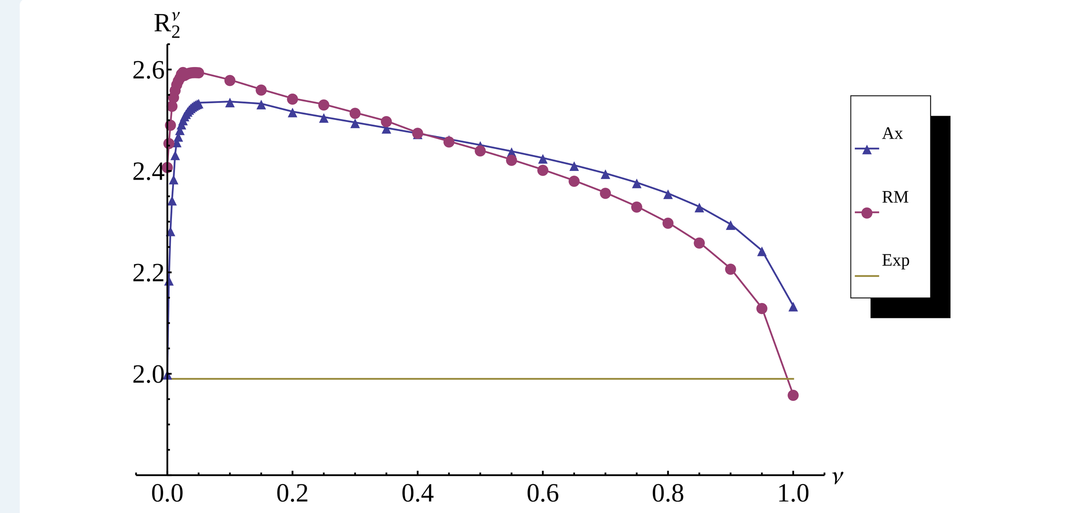


Figure: Energetic ratios of the deuterium as a function of ν for the two ansatzes considered.

We find that:

- There is a bound state at the classical level for $\nu = 1$ (dot below the $\mathrm{R}_2^\nu = 2$ limit) with the rational map ansatz but this is not new information since this case was well known for the Skyrme Model.
- A bound state is also possible if we have $\alpha \gg \beta$, $\beta \sim 10^{-8}$ (very small ν values) while adding other energetic contributions.
- Thus, further analyses of these results suggest that a minimization of the form $E_{stat} = (1-\nu)E_0 + \nu E_2 + \cancel{\nu E_4} + (1-\nu)E_6$ could lead to bound states. Indeed, we obtain the following figure.

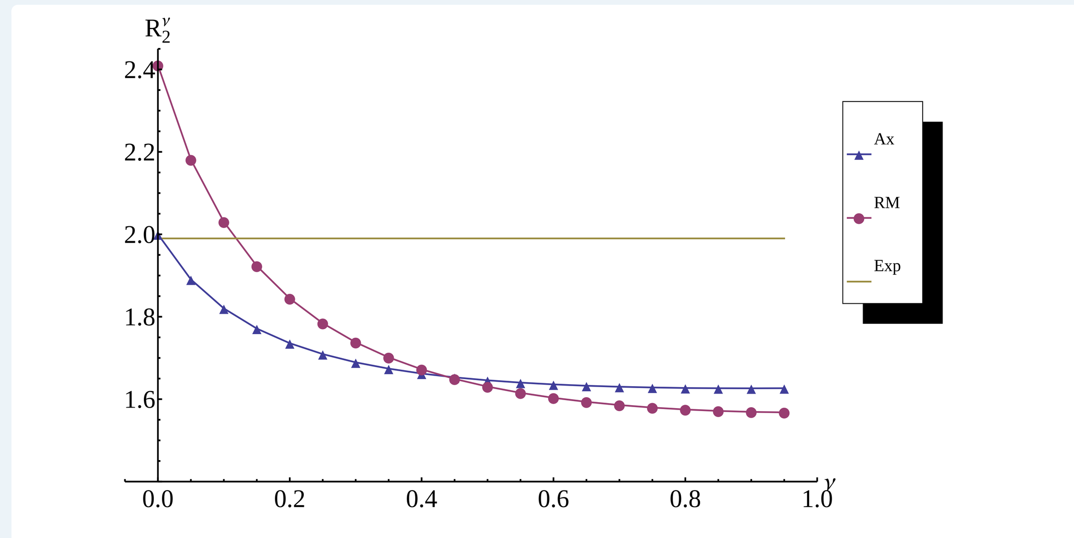


Figure: Energetic ratios of the deuterium (without the E_4 term) as a function of ν for the two ansatzes considered.

Similar figures are obtained for the case of helium-4 ($n=4$) but they are not shown here due to space constraints and the fact that they leads to similar conclusions. Further analysis of these results suggest that:

- Removing the $\mathcal{L}_4$ term in the near-BPS Lagrangian does not destabilize the soliton solutions since the $\mathcal{L}_6$ term can account for this specific task.
- Bound states (stable states) can be observed from the previous results. Moreover, in the region dominated by the BPS model ($0 \leq \nu \leq 0.5$), these bound states are best described by the axially symmetric ansatz (lower ratios).
- Provided that we set $\beta = 0$ in equations (7)-(10), the perturbative approach can be used in conjunction with the axially symmetric ansatz to describe nuclei (with the advantages of analytic solutions).
- Given our choice of potential and parameters, we find a very small pion decay constant and no pion mass

  $$F_\pi = 0.09 \text{ MeV} \qquad (F_\pi^{\text{expt}} = 186 \text{ MeV})$$
  $$m_\pi = 0 \qquad (m_\pi^{\text{expt}} = 138 \text{ MeV})$$

  so the link to soft-pion physics is yet unclear.
- The solution that minimizes the static energy (in this case the mass of the nuclei) for a given n remains unknown.
- Several aspects of the model remain to be studied (e.g. magnetic moments, vibrational and rotational excitations....).

Conclusion

The goal of this work was to find the best parametric representation of U that leads to minimal energetic configurations of the near BPS-Skyrme model. This remains an important question since the real structure of nuclei is still unknown and, as we have seen, the choices we make directly affect the resulting energy and geometry of the solutions in this model. A suitable parametrization could indeed lead to an improvement of the predictive power of the model.
It turns out that both ansatzes used in this work are not appropriate to minimize the four terms in the static energy of the near-BPS Skyrme Model simultaneously. A compromise can be reached if we remove the $\mathcal{L}_4$ term in the Lagrangian, which allow us to recover the perturbative approach which was previously assumed to describe nuclei. That is to say, approximate the lowest energy solution by an analytical axial solution, that of the BPS Model, and compute directly the relevant physical quantities.

References/Acknowledgements

[1] T.H.R. Skyrme, Proc. R. Soc. A 260, 127 (1961).
[2] T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113, 843 (2005); P. Sutcliffe, Phys. Rev. D 79, 085014 (2009).
[3] C. Adam, J. Sanchez-Guillen, A. Wereszczynski, Phys. Lett. B691, 105 (2010).
[4] M.O. Beaudoin, L. Marleau, Nucl.Phys. B883 (2014) 328-349.
[5] J.M. Speight, arXiv:1406.0739.

* This work is supported by NSERC (Canada).

email: nicolas.giasson.1@ulaval.ca / lmarleau@phy.ulaval.ca