Random Matrices Approach to Neutrino Masses

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Introduction

The origin of neutrinos masses is one of the great mysteries of modern physics. As it turns out, the Standard Model (SM) of particle physics cannot account for massive neutrinos. They can only be described as massless Weyl spinors (left handed) due to symmetry constraints. On the other hand, experiments on neutrino oscillations have provided strong evidence for non-zero masses, which indicate the presence of a mechanism or phenomenon beyond the current limit of our understanding. To generate the neutrino masses, we choose to work in a minimal extension of the SM in which three right-handed (sterile) neutrinos are added to the particle content in order to use the type I seesaw mechanism. However, due to the lack of knowledge about the underlying theory of neutrino masses and mixing, the parameters and couplings of the two mass matrices allowed by gauge invariance (i.e. the Dirac and Majorana mass matrices) cannot be predicted.

In order to overcome this obstacle, we choose to adopt the anarchy scenario proposed by Murayama et al., which allows us to randomly generate these matrices and study the resulting spectrum (for the eigenvalues) with the tools develop in the study of random matrix theory (RMT). In this work, we propose to compute the joint probability distribution for the mass matrix eigenvalues (of arbitrary dimensions) using the seesaw mechanism and extract information on neutrino masses from the resulting spectrum. This statistical analysis allows determining among other things, the hierarchy of the mass spectrum and the mass gap between generations.

Motivation

Neutrinos are only sensitive to weak interactions and gravity. As a result, to accurately describe their interactions one only needs left-handed Weyl spinor (ν_L). Consequently, the SM, being as economic as possible in its particle content, does not incorporate right-handed neutrinos (ν_R) since they are not required in any known gauge interactions (and there is no experimental evidence of their existence). This leads to important phenomenological consequences in neutrinos physics:

- Due to **gauge invariance**, **renormalization** and the **particle content**, no neutrino mass term (Dirac or Majorana) is allowed in the SM. $\Rightarrow$ Neutrinos are massless.
- The SM cannot explain neutrino oscillations (which imply massive neutrinos, see table 1).
- The mechanism by which neutrinos acquire their masses is not known. Many are suggested in the literature [4,6].

Indeed, since neutrinos **interact** with matter as weak eigenstates $|\nu_\alpha\rangle$ and **propagate** as mass eigenstates $|\nu_k\rangle$, oscillations can occur in these different basis according to the relation

$$|\nu_\alpha\rangle = \sum_{k=1}^{3} U^*_{\alpha k} |\nu_k\rangle \, ,$$

were U is known as the PMNS matrix. Experiments on neutrino oscillations have provided compelling arguments in favour of non-zero masses for the three generations.

Table 1: Values of the various oscillation parameters as of 2014 [2].

Parameters	Normal hierarchy	Inverted hierarchy
$\theta_{12}[°]$	$33.48^{+0.78}_{-0.75}$	$33.48^{+0.78}_{-0.75}$
$\theta_{13}[°]$	$8.50^{+0.20}_{-0.21}$	$8.51^{+0.20}_{-0.21}$
$\theta_{23}[°]$	$42.30^{+3.0}_{-1.6}$	$49.50^{+1.5}_{-2.2}$
$\delta[°]$	306^{+39}_{-70}	254^{+63}_{-62}
$\triangle m^2_{21}[10^{-5}\mathrm{eV}^2]$	$7.50^{+0.19}_{-0.17}$	$7.50^{+0.19}_{-0.17}$
$\triangle m^2_{31(2)}[10^{-3}\mathrm{eV}^2]$	$+2.457^{+0.047}_{-0.047}$	$-2.449^{+0.048}_{-0.047}$

The **absolute scale** of neutrino masses is not known, but upper bounds on theses values can be obtained by a variety of tests that include direct kinematic measurements, double beta decay and cosmology. These bounds are typically $\sim \mathcal{O}(\mathrm{eV})$. The need to explain this type of data leads to physics beyond the SM.

Type I seesaw mechanism

We consider a minimal extension of the SM in which three right-handed (sterile) neutrinos are added to the particle content. Under gauge invariance, two different mass terms are then allowed [4].

- **Dirac** mass term

$$\mathscr{L}_D = -\sum_{\alpha,\beta} \left(Y^{\nu}_{\alpha\beta} \overline{L_\alpha} \Phi_c \left(\nu_\beta\right)_R + Y^{\nu *}_{\beta\alpha} \overline{\left(\nu_\beta\right)}_R \Phi^\dagger_c L_\alpha \right). \tag{1}$$

- **Majorana** mass term

$$\mathscr{L}_M = -\frac{1}{2}\sum_{\alpha,\beta} \left(\overline{(\nu_\alpha)^c_R} M^R_{\alpha\beta} \left(\nu_\beta\right)_R + \overline{\left(\nu_\beta\right)_R} M^{R*}_{\beta\alpha} \left(\nu_\alpha\right)^c_R \right). \tag{2}$$

Here, $\alpha, \beta = e, \mu, \tau$, L_α is a leptonic doublet and Φ is the Higgs doublet. The matrices Y^ν and M^R correspond to the Yukawa couplings and the Majorana mass matrix (for ν_R) respectively. After **electroweak symmetry breaking**, the most general mass term takes the form [6]

$$\begin{aligned} \mathscr{L}_{D+M} &= -\overline{\boldsymbol{\nu}_L} M^D \boldsymbol{\nu}_R - \frac{1}{2}\overline{\boldsymbol{\nu}^c_R} M^R \boldsymbol{\nu}_R + \mathrm{h.c} \\ &= -\frac{1}{2} \left(\overline{\boldsymbol{\nu}_L}, \, \overline{\boldsymbol{\nu}^c_R} \right) \underbrace{\begin{pmatrix} \mathbf{0}_{3\times 3} & M^D \\ (M^D)^T & M^R \end{pmatrix}}_{M} \begin{pmatrix} \boldsymbol{\nu}^c_L \\ \boldsymbol{\nu}_R \end{pmatrix} + \mathrm{h.c}, \end{aligned} \tag{3}$$

were we have set $M^D \equiv \frac{v}{\sqrt{2}} Y^\nu$, which corresponds to the Dirac mass matrix.

Type I seesaw mechanism...

The next step is to **diagonalize** the block matrix M in order to find the physical masses and mass eigenstates. To that end, in the seesaw mechanism one makes the key assumption

$$\boxed{M^D \ll M^R}, \tag{4}$$

were the typical energy scales are chosen such that

$$M^D \sim \mathcal{O}(10^2)\mathrm{GeV}, \quad M^R \sim \mathcal{O}(10^{15})\mathrm{GeV}. \tag{5}$$

This allows for a block diagonalization to first order in the expansion parameter

$$\epsilon \equiv M^D (M^R)^{-1} \sim \mathcal{O}(10^{-13}), \tag{6}$$

so that one can write

$$U^T M U = \begin{pmatrix} M^\nu & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & M^h \end{pmatrix} + \mathcal{O}\left(\epsilon^2\right). \tag{7}$$

Note that M is a complex symmetric matrix, U is a unitary matrix and M^ν and M^h are the mass matrices of the light (active) and heavy (sterile) neutrinos. After diagonalization, these matrices are found to be given by

$$\begin{aligned} &M^\nu = -M^D (M^R)^{-1} (M^D)^T \sim \mathcal{O}(10^{-2})\mathrm{eV} \qquad (\textbf{light}), \\ &M^h \simeq M^R \qquad (\textbf{heavy}). \end{aligned} \tag{8}$$

Thus, the seesaw mechanism:

- predicts small neutrino masses for the three active neutrinos;
- requires three heavy right-handed neutrinos;
- is based on an hybrid mass term (3) of Majorana type;
- cannot explain or predict the various couplings and parameters that are contained in M^D and M^R.

Anarchy scenario and RMT

The result (8) for M^ν forms the basis on which this present work is built. In order to overcome the **lack of knowledge** on the underlying theory of neutrino masses, we adopt the anarchy scenario proposed by Murayama [3], which can be summarized in the following proposition.

Proposition 1. (Murayama *et al.*[3])
The underlying theory of nature has dynamics which produces a neutrino mass matrix which, from the viewpoint of the low-energy effective theory, displays anarchy: all entries are comparable, no pattern or structure is easily discernable, and there are no special precise ratios between any entries.

That is, we wish to impose as few constraints as possible on the free parameters of M^D and M^R. However, we need to adopt a particular framework that allows us to obtain quantitative results with the tools develop in random matrix theory. This leads to the following three hypotheses

1. There is no physical distinction among the three generations of lepton doublets (they have the **same interactions**).
2. The various couplings and mass matrix elements are randomly generated from a distribution which is **basis independent**.
3. The matrix elements are **distributed independently** from each other.

These specific constraints **uniquely determine** the joint probability density function (jpdf) for the **matrix elements** (M^D_{ij} and M^R_{ij}) with which we scan the parameter space. This is a well-known result in RMT [5] and it corresponds to the **gaussian distribution**. Due to basis independence, the resulting jpdf for the **eigenvalues** follows directly provided one can compute the measure (jacobian) associated to this **change of variables**. The jpdf $\mathcal{P}_D(M^D)dM^D$ and $\mathcal{P}_R(M^R)dM^R$ are thus given by

$$\boxed{\text{matrix elements space}} \qquad\qquad \boxed{\text{eigenvalues space}}$$

$$\underbrace{e^{-k_1 \mathrm{Tr}\left(\left(M^R\right)^\dagger M^R\right)} \prod_{i\le j} dM^R_{ij}}_{\mathcal{P}_R\left(M^R\right)dM^R} \;\rightarrow\; C_N e^{-k_1 \sum_i \mathfrak{m}_i^2} \prod_{i<j} \left|\mathfrak{m}_i^2 - \mathfrak{m}_j^2\right| \prod_{i=1}^{N} |\mathfrak{m}_i| d\mathfrak{m}_i dU,$$

$$\underbrace{e^{-k_2 \mathrm{Tr}\left(\left(M^D\right)^\dagger M^D\right)} \prod_{i,\,j} dM^D_{ij}}_{\mathcal{P}_D\left(M^D\right)dM^D} \;\rightarrow\; \tilde{C}_N e^{-k_2 \sum_i m_i^2} \prod_{i<j} \left|m_i^2 - m_j^2\right|^2 \prod_{i=1}^{N} |m_i| dm_i d\Omega,$$

with

$$d\Omega \equiv dV_1 dV_2 \left(\prod_{i=1}^{N} d\phi_i \right)^{-1},$$

where dU is the Haar measure that depends on the unitary matrix U that diagonalizes M^R according to $U^T M^R U = \mathfrak{m}_i \delta_{ij}$ and $dV_1 dV_2$ is the equivalent Haar measure for M^D. These distributions, derived entirely from a random matrix perspective, will be relevant in the next section when we discuss the neutrino mass jpdf. Thus, for all intents and purposes, the anarchy principle is the **link** between RMT and neutrino physics.

Neutrino mass distribution (jpdf)

We wish to compute $\mathcal{P}_\nu\left(M^\nu\right) dM^\nu$ and we know from (8) that M^ν is a function of M^D and M^R. Following the framework previously described, this quantity then characterizes a specific random matrix ensemble, known as the **seesaw ensemble** [1].
This ensemble is define as the family of $N \times N$ matrices of the form (8), where M^D is a complex random matrix and M^R is an invertible, symmetric and complex random matrix. The resulting distribution can be obtained by following a few steps**:**

Neutrino mass distribution (jpdf)...

Step 1: Write down the joint density for M^D and M^R assuming that they are statistically independent

$$\mathcal{P}_R(M^R)\mathcal{P}_D(M^D) dM^R dM^D. \tag{9}$$

Step 2: Introduce a M^ν dependence in (9) by an appropriate change of variables ($M^R = (M^D)^T (M^\nu)^{-1} M^D$). This gives us a joint density for M^ν and M^D.

$$\mathcal{P}_R((M^D)^T (M^\nu)^{-1} M^D) \mathcal{P}_D(M^D) \left|\det\left(\boldsymbol{J}\right)\right| dM^\nu dM^D. \tag{10}$$

Step 3: Compute the associated Jacobian $|\det(\boldsymbol{J})|$.

Step 4: Obtain $\mathcal{P}_\nu\left(M^\nu\right)$ by marginalizing (integrating) out M^D.

$$\mathcal{P}_\nu(M^\nu) = \int_{-\infty}^{\infty} \mathcal{P}_R((M^D)^T (M^\nu)^{-1} M^D) \mathcal{P}_D(M^D) \left|\det\left(\boldsymbol{J}\right)\right| dM^D. \tag{11}$$

In what follows, we limit our analysis to the case were the number of left and right handed neutrinos are equal so that M^D and M^R are square matrices of the same dimensions.

Preliminary results and analysis

From the study of the seesaw ensemble (more precisely eq. (11)) we can deduce some general properties of the distribution $\mathcal{P}_\nu(M^\nu)$.

- Under the transformation $M^\nu \to M^{\nu\prime} = U^T M^\nu U$, the particular **symmetry** $\mathcal{P}_\nu\left(M^{\nu\prime}\right) dM^{\nu\prime} = \mathcal{P}_\nu\left(M^\nu\right) dM^\nu$ implies that we can diagonalize M^ν or M^D without introducing new angular dependence in the resulting integral.
- Integration over angles and phases (contained in dM^D) can be perform since there is **no explicit angular dependence** in the integral.
- The distribution is then completely determined by an integration over the **singular values** of M^D (real and non-negative eigenvalues).
- This leads to **Selberg-type** integrals that need to be evaluated for square matrices of finite but arbitrary dimensions N.
- The resulting jpdf for M^ν will contain angles and phases which make it possible to find a **connexion** between eigenvalues and eigenvectors.

Thus, from a purely mathematical point of view, this amounts to solve multidimensional integrals of the form

$$\begin{aligned} I(t_{ij}) = \int_0^\infty \cdots \int_0^\infty \prod_{i<j} \left|x_i - x_j\right|^2 e^{-2k_2 x_i t_{ij} x_j} \cdots \\ \cdots \prod_{k=1}^{N} e^{-k_2 x_k (t_{kk} x_k + 1)} \left|x_k\right|^{N+1} dx_k, \end{aligned} \tag{12}$$

were k_2 is a real and positive constant, t_{ij} is a real and symmetric random matrix which is itself a function of M^ν and the variables x_k are related to the singular values of M^D. We are interested in the complete analytical solution of (12) since $\mathcal{P}_\nu\left(M^\nu\right) dM^\nu = f(M^\nu) I(t_{ij}) dM^\nu$.

To summarize our progress so far, a few comments are in order:

- Some simple cases can be solve completely, that is, the case where $N = 1, 2$. The result contains the hypergeometric function ${}_1F_1$.
- The case $N = 3$ is still under investigation.
- The connexion with orthogonal polynomials allows us to solve for the specific case $t_{ij} = \mathrm{diag}(a, a, ..., a) \; \forall N$, where a is a positive constant.

Conclusion

The goal of this work is to compute the mass distribution of active neutrinos using type I seesaw mechanism in the context of a random matrix theory. Having succeeded in formulating the problem in a purely mathematical fashion, we are able to solve the corresponding integral for some simple cases with no obvious physical connexion, but nonetheless instructive from a mathematical standpoint.
The physical solution $N = 3$ is still under investigation and a lot of work remains to be done. Once this major step is achieved, one can in principle make predictions on the mass gap between generations by computing correlation functions as well as predictions on the various oscillation parameters.

References/Acknowledgements

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